A NOTE ON CES FUNCTIONS

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Functions featuring constant elasticity of substitution (CES) are widely used in applied economics and finance. In this note, I do two things. First, I derive a number of conditions (such as the optimal demand schedule) when aggregation technology is CES. Second, I show how the CES function nests some particular functional forms as special cases.

1 PRELIMINARIES

Consider a general CES function:

$$Y = A\left(\sum_{j=1}^{\mathcal{J}} \alpha_j^{\frac{1}{\eta}} Y_j^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}, \tag{1}$$

where $A, \eta > 0$, $\alpha_j, Y_j > 0 \;\forall\; j$, and $\sum_{j=1}^{\mathcal{J}} \alpha_j = 1$. In models with monopolistic competition between firms, Y is interpreted as aggregate output and $\{Y_j\}_{j=1}^{\mathcal{J}}$ as output by intermediate firms. In turn, the production technology of firms can also have a CES structure, with each Y_j being referred to as an input factor such as labor or capital or land. Y can also be aggregate demand or consumption, a composite of individual consumer classes or goods. In open economy macroeconomics, one uses the CES structure to distinguish between domestically produced goods and imports. The scaling parameter A is often normalized to 1, but in the context of production theory, it is typically interpreted as a measure of total factor productivity (which can be stochastic in a time series context).

2 IMPLICATIONS OF CES TECHNOLOGY

What are the implications of assuming a CES technology for aggregation in economic models? To shed light on this question, we start with a maximization problem. The problem is to choose the vector $\{Y_j\}_{j=1}^{\mathcal{J}}$ that maximizes Y subject to some budget constraint

$$\sum_{j=1}^{\mathcal{J}} P_j Y_j \leq Z, \tag{2}$$

where Z is total money spent. Let us state the Lagrangian (where Λ is the multiplier for the constraint):

$$\mathcal{L} = A\left(\sum_{j=1}^{\mathcal{J}} \alpha_j^{\frac{1}{\eta}} Y_j^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} - \Lambda\left(\sum_{j=1}^{\mathcal{J}} P_j Y_j - Z\right)$$

The necessary first order condition is $A^{\frac{\eta-1}{\eta}} Y^{\frac{1}{\eta}} \alpha_j^{\frac{1}{\eta}} Y_j^{-\frac{1}{\eta}} = \Lambda P_j$. This must hold for all $j \in \mathcal{J}$, so we can write

$$Y_j = \frac{\alpha_j}{\alpha_i}\left(\frac{P_j}{P_i}\right)^{-\eta} Y_i \;\;\forall\;\; i, j \tag{3}$$

2.1 A NATURAL PRICE INDEX

Next, we derive an exact price index for a unit increase in Y. Plug (3) into the budget constraint and solve for Y_i:

$$
\begin{aligned}
Z &= \sum_{j=1}^{\mathcal{J}} P_j Y_j = \frac{Y_i P_i^{\eta}}{\alpha_i} \sum_{j=1}^{\mathcal{J}} \alpha_j P_j^{1-\eta} \\
\Rightarrow Y_i &= \frac{Z \alpha_i P_i^{-\eta}}{\sum_{j=1}^{\mathcal{J}} \alpha_j P_j^{1-\eta}}
\end{aligned}
$$

Insert this result into Y (now with indexing i):

$$
\begin{aligned}
Y &= A \left(\sum_{i=1}^{\mathcal{J}} \alpha_i^{\frac{1}{\eta}} Y_i^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} \\
&= A \left(\sum_{i=1}^{\mathcal{J}} \alpha_i^{\frac{1}{\eta}} \left(\frac{Z \alpha_i P_i^{-\eta}}{\sum_{j=1}^{\mathcal{J}} \alpha_j P_j^{1-\eta}} \right)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} \\
&= AZ \left(\sum_{j=1}^{\mathcal{J}} \alpha_j P_j^{1-\eta} \right)^{-1} \left(\sum_{i=1}^{\mathcal{J}} \alpha_i P_i^{1-\eta} \right)^{\frac{\eta}{\eta-1}} \\
&= AZ \left(\sum_{j=1}^{\mathcal{J}} \alpha_j P_j^{1-\eta} \right)^{\frac{1}{\eta-1}}
\end{aligned}
$$

Finally, define P as the expenditure needed to buy one unit of Y, i.e. $P \equiv Z|_{Y=1}$. From the expression above, it naturally follows that

$$
P = \frac{1}{A} \left(\sum_{j=1}^{\mathcal{J}} \alpha_j P_j^{1-\eta} \right)^{\frac{1}{1-\eta}}. \tag{4}
$$

Thus, P is the natural price index for Y.

2.2 THE BUDGET CONSTRAINT REVISITED

Next, note that inserting (3) into (2) gives

$$
\begin{aligned}
Z &= \sum_{j=1}^{\mathcal{J}} P_j Y_j \\
&= \sum_{j=1}^{\mathcal{J}} P_j \frac{\alpha_j}{\alpha_i} \left(\frac{P_j}{P_i} \right)^{-\eta} Y_i \\
&= P_i^{\eta} \frac{Y_i}{\alpha_i} A^{1-\eta} P^{1-\eta} \\
\Rightarrow Y_i &= \alpha_i \left(\frac{P_i}{P} \right)^{-\eta} \frac{Z}{A^{1-\eta} P}.
\end{aligned} \tag{5}
$$

Substitute into (1) and solve for Z. The result is

$$Y = A\left(\sum_{j=1}^{\mathcal{J}} \alpha_j^{\frac{1}{\eta}}\left(\alpha_j\left(\frac{P_j}{P}\right)^{-\eta}\frac{Z}{A^{1-\eta}P}\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} = ZP^{-1}.$$

Thus, $Z = PY$, and we can write the budget constraint as

$$PY = \sum_{j=1}^{\mathcal{J}} P_j Y_j.$$

2.3 The optimal demand schedule

Finally, substitution of $Z = PY$ into (5) (which holds $\forall\ i, j \in \mathcal{J}$) gives the optimal demand for Y_j:

$$Y_j = \alpha_j\left(\frac{P_j}{P}\right)^{-\eta}\frac{Y}{A^{1-\eta}} \tag{6}$$

One can also get rid of A and end up with

$$Y_j = \alpha_j\left(\frac{P_j}{\left(\sum_{i=1}^{\mathcal{J}}\alpha_i P_i^{1-\eta}\right)^{\frac{1}{1-\eta}}}\right)^{-\eta}\left(\sum_{i=1}^{\mathcal{J}}\alpha_i^{\frac{1}{\eta}}Y_i^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}.$$

Thus, optimal demand for Y_j in (6) is not determined by A, but rather by the price P_j relative to all other prices, as well as by total factor demand. Equation (6) is widely used as demand function in theoretical models of optimizing economic behavior.

2.4 Elasticities

To understand why the functional form in (1) exhibits CES between any pair Y_i, Y_j, we define the elasticity of substitution as the percentage rise in $\frac{Y_j}{Y_i}$ following a one percent rise in the relative price $\frac{P_i}{P_j}$. It is clear from (3) that $\frac{Y_j}{Y_i} = \frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta}$, so we can write

$$\frac{\partial\left(\frac{Y_j}{Y_i}\right)}{\partial\left(\frac{P_i}{P_j}\right)} = \eta\frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta-1}.$$

By dividing by $\left(\frac{Y_j}{Y_i}\right)/\left(\frac{P_i}{P_j}\right)$, a constant elasticity of substitution emerges:

$$\text{Elasticity of substitution} \equiv \frac{\partial\left(\frac{Y_j}{Y_i}\right)/\left(\frac{Y_j}{Y_i}\right)}{\partial\left(\frac{P_i}{P_j}\right)/\left(\frac{P_i}{P_j}\right)} = \frac{\eta\frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta-1}}{\frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta-1}} = \eta$$

What about the income elasticity? Earlier we defined Z as the income spent on Y. The income elasticity follows from (5):

$$\text{Elasticity of income} \equiv \frac{\partial Y_j/Y_j}{\partial Z/Z} = \frac{\alpha_i\left(\frac{P_i}{P}\right)^{-\eta}\frac{1}{A^{1-\eta}P}}{\alpha_i\left(\frac{P_i}{P}\right)^{-\eta}\frac{1}{A^{1-\eta}P}} = 1$$

3 Special cases of the CES function

Next, we show how the CES function nests as special cases some particular functional forms. Before we start, it is useful to note that equation (1) is equivalent to

$$Y = A\left(\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}\right)^{\frac{1}{\gamma}}, \tag{7}$$

where $\tilde{Y}_j \equiv \frac{Y_j}{\alpha_j}$ and $\gamma \equiv \frac{\eta-1}{\eta}$. Our task is to find an expression for Y when i) $\eta \to \infty$ or $\gamma \to 1$ (linear), ii) $\eta \to 1$ or $\gamma \to 0$ (Cobb-Douglas), and iii) $\eta \to 0$ or $\gamma \to -\infty$ (Leontief). For reasons that will be clear below, it is convenient to work with a log transformation of equation (7):

$$\ln(Y) = \ln(A) + \frac{1}{\gamma}\ln\left(\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}\right) = \ln(A) + \frac{F(\gamma)}{G(\gamma)}, \tag{8}$$

where $F(\gamma) \equiv \ln\left(\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}\right)$ and $G(\gamma) \equiv \gamma$. Finally, in order to derive the Cobb-Douglas and Leontief functions we need L'Hôpital's rule:

The L'Hôpital's rule:

Consider two functions $F(\gamma)$ and $G(\gamma)$, which are differentiable on an interval (a, b), except possibly at a point $c \in (a, b)$. Suppose $\lim_{\gamma\to c} F(\gamma) = \lim_{\gamma\to c} G(\gamma) = 0$ or $\pm\infty$, $\lim_{\gamma\to c}\frac{F'(\gamma)}{G'(\gamma)}$ exists, and $G'(\gamma) \neq 0\ \forall\ \gamma \neq c \in (a, b)$. Then

$$\lim_{\gamma\to c}\frac{F(\gamma)}{G(\gamma)} = \lim_{\gamma\to c}\frac{F'(\gamma)}{G'(\gamma)}.$$

In our case the first derivatives are $F'(\gamma) = \frac{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}\ln(\tilde{Y}_j)}{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}}$ and $G'(\gamma) = 1$, implying

$$\frac{F'(\gamma)}{G'(\gamma)} = \frac{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}\ln\left(\tilde{Y}_j\right)}{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}}. \tag{9}$$

Next, we consider each of the three special cases of the CES function.

3.1 The linear case ($\gamma \to 1$)

This one is trivial as it follows immediately from equation (7) that

$$\lim_{\gamma\to 1} Y = A\sum_{j=1}^{\mathcal{J}} Y_j. \tag{10}$$

3.2 The Cobb-Douglas case ($\gamma \to 0$)

Here we want to evaluate equation (8) when $\gamma \to 0$. However, the result is

$$\lim_{\gamma\to 0} \ln(Y) = \ln(A) + \lim_{\gamma\to 0} \frac{\ln\left(\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}\right)}{\gamma} = \ln(A) + \text{“}\left(\frac{0}{0}\right)\text{”}.$$

The last term holds as long as $\sum_{j=1}^{\mathcal{J}} \alpha_j = 1$.[1] To proceed, we apply L'Hôpital's rule. Evaluating equation (9) in the limit as $\gamma \to 0$, we can write (8) as

$$\begin{aligned}\lim_{\gamma\to 0} \ln(Y) &= \ln(A) + \lim_{\gamma\to 0} \frac{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma} \ln\left(\tilde{Y}_j\right)}{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}} \\ &= \ln(A) + \sum_{j=1}^{\mathcal{J}} \alpha_j \ln\left(\tilde{Y}_j\right) \\ &= \ln\left(A \prod_{j=1}^{\mathcal{J}} \tilde{Y}_j^{\alpha_j}\right),\end{aligned}$$

or

$$\lim_{\gamma\to 0} Y = A \frac{1}{\prod_{j=1}^{\mathcal{J}} \alpha_j^{\alpha_j}} \prod_{j=1}^{\mathcal{J}} Y_j^{\alpha_j}. \tag{11}$$

This is the widely used Cobb-Douglas function. The scaling parameter A is often normalized so that $A \frac{1}{\prod_{j=1}^{\mathcal{J}} \alpha_j^{\alpha_j}} = 1$.

3.3 The Leontief case ($\gamma \to -\infty$)

When $\gamma \to -\infty$, then equation (8) becomes

$$\lim_{\gamma\to -\infty} \ln(Y) = \ln(A) + \lim_{\gamma\to -\infty} \frac{\ln\left(\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}\right)}{\gamma} = \ln(A) + \text{“}\left(\frac{-\infty}{-\infty}\right)\text{”}.$$

Again, using L'Hôpital's rule, we can write

$$\lim_{\gamma\to -\infty} \ln(Y) = \ln(A) + \lim_{\gamma\to -\infty} \frac{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma} \ln\left(\tilde{Y}_j\right)}{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}}.$$

However, a new problem arises because $\lim_{\gamma\to -\infty} \frac{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma} \ln\left(\tilde{Y}_j\right)}{\sum_{j=1}^{\mathcal{J}} \alpha_j \tilde{Y}_j^{\gamma}} = \text{“}\left(\frac{0}{0}\right)\text{”}$, which cannot be solved by multiple uses of L'Hôpital's rule.[2] In order to proceed, we define $\tilde{Y}_m \equiv \min\left\{\tilde{Y}_1, \ldots, \tilde{Y}_{\mathcal{J}}\right\}$. Dividing the nominator and denominator by $\tilde{Y}_m^{\gamma}$, we get

$$\lim_{\gamma\to -\infty} \ln(Y) = \ln(A) + \lim_{\gamma\to -\infty} \frac{\sum_{j=1}^{\mathcal{J}} \alpha_j \left(\frac{\tilde{Y}_j}{\tilde{Y}_m}\right)^{\gamma} \ln\left(\tilde{Y}_j\right)}{\sum_{j=1}^{\mathcal{J}} \alpha_j \left(\frac{\tilde{Y}_j}{\tilde{Y}_m}\right)^{\gamma}}$$

[1] If instead $\sum_{j=1}^{\mathcal{J}} \alpha_j > 1$ (< 1), then the last term goes to ∞ ($-\infty$) as $\gamma \to 0$.

[2] $\tilde{Y}_j^{\gamma}$ will always show up both in the nominator and in the denominator.

$$
\begin{aligned}
&= \ln(A) + \ln\left(\min\left\{\tilde{Y}_1, \dots, \tilde{Y}_{\mathcal{J}}\right\}\right) \\
&= \ln\left(A \min\left\{\tilde{Y}_1, \dots, \tilde{Y}_{\mathcal{J}}\right\}\right),
\end{aligned}
$$

or

$$
\lim_{\gamma \to -\infty} Y = A \min\left\{\frac{Y_1}{\alpha_1}, \dots, \frac{Y_{\mathcal{J}}}{\alpha_{\mathcal{J}}}\right\}. \tag{12}
$$

This is the Leontief function.

3.4 From isoelastic utility to log utility

Consider the isoelastic utility function

$$
U = \frac{C^{1-\sigma} - 1}{1 - \sigma},
$$

with $C > 0$. Apparently a " $\left(\frac{0}{0}\right)$ "-issue arises when one evaluates U as $\sigma \to 1$. In that case one can define $F(\sigma) \equiv C^{1-\sigma} - 1$ and $G(\sigma) \equiv 1 - \sigma$. Because $F'(\sigma) = -C^{1-\sigma} \ln(C)$ and $G'(\sigma) = -1$, we get

$$
\lim_{\sigma \to 1} U = \lim_{\sigma \to 1} \frac{F'(\sigma)}{G'(\sigma)} = \ln(C). \tag{13}
$$