An Introduction to Labor Market Search

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1. Introduction

These notes replicate the review paper by Rogerson, Shimer and Wright (2005). Central to the search theory approach is the notion that trading frictions are important. It takes time and other resources for a worker to land a job, especially a good job at a good wage, and for a firm to fill a vacancy. There is simply no such thing as a centralized market where buyers and sellers of labor meet and trade at single price, as assumed in classical equilibrium theory. Two questions are paramount: First, how do agents meet? In particular, is search random, so that unemployed workers are equally likely to locate any job opening, or directed, so that for example firms can attract more applicants by offering higher wages? Second, exactly how are wages determined? Do matched workers and firms bargain, or are wages posted unilaterally before they meet? Section 2 begin with the problem of a single agent looking for a job. This simple model is a building block for the equilibrium analysis to follow. Section 3 presents some generalizations designed to help understand turnover and labor market transitions. This section also introduces tools and techniques needed for equilibrium search theory. Section 4 establishes a class of equilibrium models built on two main ingredients: the matching function and the bargaining solution. In section 5, we consider an environment where wages are posted ex ante, rather than bargained after agents meet, and in addition where search is directed, i.e. workers do not encounter firms completely at random but try to locate those posting attractive terms of trade. Models where wages are posted but search is once again purely random are studied in section 6. Section 7 analyzes efficiency.

2. A Simple Job Search Model

2.1 Discrete Time

Consider an individual searching for a job in discrete time, taking market conditions as given. He seeks to maximize:

$$E_0\{\sum_{t=0}^{\infty} \beta^t x_t\}$$

where $\beta \in (0,1)$ is the discount factor, x_t is period t income, and $x = w$ if the individual is employed and $x = b$ if the individual is unemployed. w is the wage and b is the unemployment benefit. Suppose that in each period the individual gets a job offer with a wage drawn from a known i.i.d. distribution $F(w)$. Assume that if a job is accepted, it is kept forever. The Bellman equations for being employed and unemployed are given by:

$$W(w) = w + \beta W(w) \quad (1)$$

$U = b + \beta \int_0^{\infty} max\{U, W(w)\}\, dF(w)$ (2)

where $W(w)$ is the payoff from accepting a wage w and U is the payoff from rejecting a wage offer, earning b, and sampling again next period. Note also that $dF(w) = f(w)dw$, where $f(w)$ is the density of w. Solving (2) for $W(w)$ we get:

$$W(w) = \frac{w}{1-\beta}$$

Since $W(w)$ is strictly increasing in w, there is a unique reservation wage w_R such that:

$$W(w_R) = U = \frac{w_R}{1-\beta}$$

with the property that the worker should reject $w < w_R$ and accept $w \geq w_R$. The reservation wage must satisfy (2), and we can write:

$$\frac{w_R}{1-\beta} = b + \beta \int_0^{\infty} max\left\{\frac{w_R}{1-\beta}, \frac{w}{1-\beta}\right\} dF(w)$$

$$\Rightarrow \quad w_R = (1-\beta)b + \beta \int_0^{\infty} max\{w_R, w\}\, dF(w) \qquad (3)$$

The optimal search strategy is completely characterized by (3), but sometimes it is convenient to write it in another way. Subtracting βw_R on both sides we get:

$$(1-\beta)w_R = (1-\beta)b + \beta \int_0^{\infty} max\{w_R, w\}\, dF(w) - \beta w_R = (1-\beta)b + \beta \int_0^{\infty} max\{w_R - w_R, w - w_R\}\, dF(w)$$

$$= (1-\beta)b + \beta \int_0^{\infty} max\{0, w - w_R\}\, dF(w) = (1-\beta)b + \beta \int_{w_R}^{\infty} (w - w_R) dF(w)$$

$$\Rightarrow \quad w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\infty} (w - w_R) dF(w) \qquad (4)$$

Using integration by parts, we can set up yet another representation of the reservation wage. Remember how integration by parts work. For two generic variables u and v:

$$(uv)' = u'v + uv'$$

$$\Rightarrow \quad \int (uv)' = \int u'v + \int uv'$$

$$\Rightarrow \quad \int uv' = uv - \int u'v$$

Let $u = w - w_R$ and $v' = f(w)$, implying that $u' = 1$ and $v = F(w)$. Then (4) becomes:

$$w_R = b + \frac{\beta}{1-\beta}\int_{w_R}^{\infty}(w - w_R)f(w)dw = b + \frac{\beta}{1-\beta}\left[|(w - w_R)F(w)|_{w_R}^{\infty} - \int_{w_R}^{\infty}F(w)dw\right]$$

$$= b + \frac{\beta}{1-\beta}\left[(\infty - w_R)F(\infty) - (w_R - w_R)F(w_R) - \int_{w_R}^{\infty}F(w)dw\right]$$

$$= b + \frac{\beta}{1-\beta}\left[(\infty - w_R)1 - \int_{w_R}^{\infty}F(w)dw\right]$$

$$\Rightarrow \quad w_R = b + \frac{\beta}{1-\beta}\int_{w_R}^{\infty}[1 - F(w)]dw \tag{5}$$

2.2 Continuous Time

In the continuous time model risk neutral workers maximize:

$$\int_0^{\infty} e^{-rt}x(t)dt$$

where $x(t) = w$ if employed and $x(t) = b$ if unemployed. The transformation of the value functions from discrete to continuous time is as follows: First, generalize the discrete time model to allow the length of a period to be Δ. Let $\beta \equiv \frac{1}{1+r\Delta}$ and assume that the worker gets a wage offer with probability $\alpha\Delta$ in each period. Then the payoffs to working and unemployment satisfy the following versions of (1) and (2):

$$W(w) = \Delta w + \frac{1}{1+r\Delta}W(w) \tag{6}$$

$$U = \Delta b + \frac{\alpha\Delta}{1+r\Delta}\int_0^{\infty} max\{U, W(w)\}\, dF(w) + \frac{1-\alpha\Delta}{1+r\Delta}U \tag{7}$$

Rewriting (6) we get:

$$\frac{1+r\Delta-1}{1+r\Delta}W(w) = \Delta w$$

$$\Rightarrow \quad r\Delta W(w) = (1 + r\Delta)\Delta w$$

$$\Rightarrow \quad rW(w) = (1 + r\Delta)w \tag{8}$$

Rewriting (7) we get:

$$(1 + r\Delta)U = (1 + r\Delta)\Delta b + \alpha\Delta\int_0^{\infty} max\{U, W(w)\}\, dF(w) + (1 - \alpha\Delta)U$$

$$\Rightarrow \quad r\Delta U = (1 + r\Delta)\Delta b + \alpha\Delta\int_0^{\infty} max\{U, W(w)\}\, dF(w) - \alpha\Delta U$$

$$\Rightarrow \quad rU = (1 + r\Delta)b + \alpha\int_0^{\infty} max\{U, W(w)\}\, dF(w) - \alpha U$$

$$\Rightarrow \quad rU = (1 + r\Delta)b + \alpha\int_0^{\infty} max\{U - U, W(w) - U\}\, dF(w)$$

$$\Rightarrow \quad rU = (1 + r\Delta)b + \alpha\int_0^{\infty} max\{0, W(w) - U\}\, dF(w) \tag{9}$$

When $\Delta \rightarrow 0$ we obtain the continuous time Bellman equations from (8) and (9):

$$rW(w) = w \tag{10}$$

$$rU = b + \alpha \int_0^\infty max\{0, W(w) - U\}\, dF(w) \tag{11}$$

Intuitively, while U is the value of being unemployed, rU is the flow (per period) value. This equals the sum of the instantaneous payoff b, plus the expected value of any changes in the value of the worker's state, which in this case is the probability that he gets an offer α, times the expected increase in value associated with the offer, noting that the offer can be rejected. A couple of results are obtained from (10) and the condition $W(w_R) = U$:

$$W(w) = \frac{w}{r}$$

$$W(w_R) = U = \frac{w_R}{r}$$

Thus, $W(w) - U = \frac{w - w_R}{r}$ and (11) can be written:

$$w_R = b + \alpha \int_0^\infty max\left\{0, \frac{w - w_R}{r}\right\} dF(w)$$

$$\Rightarrow \quad w_R = b + \frac{\alpha}{r} \int_{w_R}^\infty (w - w_R) dF(w) \tag{12}$$

Again one can integrate by parts:

$$w_R = b + \frac{\alpha}{r} \int_{w_R}^{\infty} (w - w_R) f(w) dw = b + \frac{\alpha}{r} \left[|(w - w_R)F(w)|_{w_R}^{\infty} - \int_{w_R}^{\infty} F(w) dw \right]$$

$$= b + \frac{\alpha}{r} \left[(\infty - w_R)F(\infty) - (w_R - w_R)F(w_R) - \int_{w_R}^{\infty} F(w) dw \right]$$

$$= b + \frac{\alpha}{r} \left[(\infty - w_R)F(\infty) - \int_{w_R}^{\infty} F(w) dw \right]$$

$$\Rightarrow \quad w_R = b + \frac{\alpha}{r} \int_{w_R}^\infty [1 - F(w)] dw \tag{13}$$

Sometimes it is of interest to endogenize the search intensity. Suppose a worker can affect the arrival rate of offers α, at a cost $g(\alpha)$, where $g' > 0$ and $g'' > 0$. The Bellman equation for employed workers is unchanged, but unemployed workers now maximize $rU = w_R$ with respect to α, subject to an augmented version of (12):

$$w_R = b + \frac{\alpha}{r} \int_{w_R}^\infty (w - w_R) dF(w) - g(\alpha) \tag{14}$$

The first order condition for an interior solution is:

$$\frac{\partial rU}{\partial \alpha} = \frac{1}{r}\int_{w_R}^{\infty}(w - w_R)dF(w) - g'(\alpha) = 0$$

$$\Rightarrow \quad \int_{w_R}^{\infty}(w - w_R)dF(w) = rg'(\alpha) \qquad (15)$$

Worker behavior is then characterized by a pair (w_R, α) solving (14) and (15). It is straight forward to show that an increase in b, e.g., raises w_R and reduces α.

2.3 The average duration of an unemployment spell

Suppose the probability that a worker has not found a job after a spell of length t is e^{-Ht}, where:

$$H = \alpha[1 - F(w_R)]$$

is called the hazard rate and equals the product of the contact rate α and the probability of accepting $1 - F(w_R)$. The average duration spell D is then:

$$D = \int_0^{\infty} tHe^{-Ht}dt$$

This can be simplified using integration by parts. Remember $\int uv' = uv - \int u'v$. Let $u = t$ and $v' = He^{-Ht}$, implying that $u' = 1$ and $v = -e^{-Ht}$. Then the average duration of an unemployment spell is:

$$D = \int_0^{\infty} tHe^{-Ht}dt = |-te^{-Ht}|_0^{\infty} - \int_0^{\infty} -e^{-Ht}dt = |-te^{-Ht}|_0^{\infty} - \left|\frac{1}{H}e^{-Ht}\right|_0^{\infty}$$

$$= -\infty e^{-H\infty} + 0e^{0} - \frac{1}{H}e^{-H\infty} + \frac{1}{H}e^{-0}$$

$$\Rightarrow \quad D = \frac{1}{H} \qquad (16)$$

Also, the observed distribution of wages is $G(w) = F(w|w \geq w_R)$. Consider the impact of an increase in b, assuming for simplicity that search intensity and hence α are fixed. From (12), the immediate effect is to increase w_R, which has two secondary effects: the distribution of observed wages is higher in the sense of first order stochastic dominance, since more low wage offers are rejected; and the hazard rate H is lower, which increases average unemployment duration.

3. Worker Turnover

3.1 Transitions to Unemployment

We will now extend the basic model to include transitions to unemployment. Assume that workers face a layoff risk according to a Poisson process with parameter λ, which is an exogenous constant for now.

Introducing exogenous separations does not affect the Bellman equation for U, which is still given by (11). The Bellman equation for $W(w)$ changes however:

$$rW(w) = w + \lambda[U - W(w)] \tag{17}$$

Solving (17) for $W(w)$ yields:

$$(r+\lambda)W(w) = w + \lambda U$$
$$\Rightarrow \quad W(w) = \frac{w+\lambda U}{r+\lambda}$$

Thus, the relationship between U and the reservation wage is:

$$W(w_R) = U = \frac{w_R+\lambda U}{r+\lambda}$$
$$\Rightarrow \quad \frac{r+\lambda-\lambda}{r+\lambda}U = \frac{w_R}{r+\lambda}$$
$$\Rightarrow \quad U = \frac{w_R}{r}$$

Inserting the results into (11) gives:

$$w_R = b + \alpha\int_0^\infty max\{0, W(w) - U\}\, dF(w) = b + \alpha\int_0^\infty max\left\{0, \frac{w+\lambda U}{r+\lambda} - U\right\} dF(w)$$
$$= b + \alpha\int_0^\infty max\left\{0, \frac{w}{r+\lambda} - \frac{rU}{r+\lambda}\right\} dF(w) = b + \alpha\int_0^\infty max\left\{0, \frac{w}{r+\lambda} - \frac{w_R}{r+\lambda}\right\} dF(w)$$
$$= b + \frac{\alpha}{r+\lambda}\int_0^\infty max\{0, w - w_R\}\, dF(w) = b + \frac{\alpha}{r+\lambda}\int_{w_R}^\infty (w - w_R) dF(w)$$

Integration by parts gives:

$$w_R = b + \frac{\alpha}{r+\lambda}\int_{w_R}^\infty (w - w_R) f(w) dw = b + \frac{\alpha}{r+\lambda}\left[|(w - w_R)F(w)|_{w_R}^\infty - \int_{w_R}^\infty F(w) dw\right]$$
$$= b + \frac{\alpha}{r+\lambda}\left[(\infty - w_R)F(\infty) - (w_R - w_R)F(w_R) - \int_{w_R}^\infty F(w) dw\right]$$

$$\Rightarrow \quad w_R = b + \frac{\alpha}{r+\lambda}\int_{w_R}^\infty [1 - F(w)] dw \tag{18}$$

Thus, λ affects w_R only by changing the effective discount rate to $r + \lambda$. However, a worker now goes through repeated spells of employment and unemployment. When unemployed, he gets a job at rate $H = \alpha[1 - F(w_R)]$. While employed, he loses the job at rate λ. It is possible to endogenize λ. Suppose w

can change at a given job according to a Poisson process λ, and that in the event of a wage change a new w' is drawn from $F(w'|w)$. When the wage changes the worker can stay employed at w' or quit to unemployment. The exogenous layoff model is a special case where w' with probability 1, so that at rate λ the job effectively disappears. In this more general model, the Bellman equation for having a job becomes:

$$rW(w) = w + \lambda \int_0^\infty max\{W(w') - W(w), U - W(w)\}\, dF(w'|w) \tag{19}$$

We will consider the simplest case when $F(w'|w) = F(w)$. To solve (19) for $W(w)$ we first have to solve (11) for the integral:

$$rU = b + \alpha \int_0^\infty max\{0, W(w) - U\}\, dF(w) = b + \alpha \int_{w_R}^\infty [W(w) - U] dF(w)$$

$$\Rightarrow \quad \int_{w_R}^\infty [W(w) - U] dF(w) = \frac{rU-b}{\alpha}$$

Then we can solve (19) for $W(w)$. Take it out of the integral, and at the same time add and subtract λU. Finally, use the result above:

$$rW(w) = w - \lambda W(w) + \lambda \int_0^\infty max\{W(w'), U\}\, dF(w'|w) - \lambda U + \lambda U$$

$$= w - \lambda W(w) + \lambda \int_0^\infty max\{W(w') - U, 0\}\, dF(w'|w) + \lambda U$$

$$= w - \lambda W(w) + \lambda \int_{w_R}^\infty [W(w') - U] dF(w'|w) + \lambda U = w - \lambda W(w) + \lambda \frac{rU - b}{\alpha} + \lambda U$$

$$\Rightarrow \quad (r + \lambda) W(w) = w + \lambda \frac{rU-b}{\alpha} + \lambda U$$

$$\Rightarrow \quad W(w) = \frac{w}{r+\lambda} + \frac{\lambda}{\alpha} \frac{rU-b}{r+\lambda} + \frac{\lambda U}{r+\lambda}$$

Evaluate this equation when $w = w_R$, i.e. when $W(w_R) = U$:

$$U = \frac{w_R}{r+\lambda} + \frac{\lambda}{\alpha} \frac{rU-b}{r+\lambda} + \frac{\lambda U}{r+\lambda}$$

$$\Rightarrow \quad \left(r + \lambda - \frac{\lambda r}{\alpha} - \lambda\right) U = w_R - \frac{\lambda b}{\alpha}$$

$$\Rightarrow \quad \frac{\alpha-\lambda}{\alpha} rU = w_R - \frac{\lambda b}{\alpha}$$

$$\Rightarrow \quad U = \frac{\alpha w_R}{r(\alpha-\lambda)} - \frac{\lambda b}{r(\alpha-\lambda)}$$

Now we have a functional expression for U that will be useful later. To get an expression for $W(w)$, we combine the two results above:

$$W(w) = \frac{w}{r+\lambda} + \frac{\lambda}{\alpha}\frac{rU-b}{r+\lambda} + \frac{\lambda U}{r+\lambda} = \frac{w}{r+\lambda} + \frac{\lambda}{\alpha}\frac{\frac{\alpha w_R}{\alpha-\lambda} - \frac{\lambda b}{\alpha-\lambda} - b}{r+\lambda} + \frac{\lambda\left[\frac{\alpha w_R}{r(\alpha-\lambda)} - \frac{\lambda b}{r(\alpha-\lambda)}\right]}{r+\lambda}$$

$$= \frac{w}{r+\lambda} + \frac{\lambda}{\alpha}\frac{\frac{\alpha w_R}{\alpha-\lambda} - \frac{\lambda+\alpha-\lambda}{\alpha-\lambda}b}{r+\lambda} + \frac{\frac{\alpha\lambda w_R}{r(\alpha-\lambda)} - \frac{\lambda^2 b}{r(\alpha-\lambda)}}{r+\lambda}$$

$$= \frac{w}{r+\lambda} + \frac{\lambda w_R}{(\alpha-\lambda)(r+\lambda)} - \frac{\lambda b}{(\alpha-\lambda)(r+\lambda)} + \frac{\alpha\lambda w_R}{r(\alpha-\lambda)(r+\lambda)} - \frac{\lambda^2 b}{r(\alpha-\lambda)(r+\lambda)}$$

Finally, we derive an expression for $W(w) - U$ in the integral in (11):

$$W(w) - U = \frac{w}{r+\lambda} + \frac{\lambda w_R}{(\alpha-\lambda)(r+\lambda)} - \frac{\lambda b}{(\alpha-\lambda)(r+\lambda)} + \frac{\alpha\lambda w_R}{r(\alpha-\lambda)(r+\lambda)} - \frac{\lambda^2 b}{r(\alpha-\lambda)(r+\lambda)} - \frac{\alpha w_R}{r(\alpha-\lambda)}$$

$$+ \frac{\lambda b}{r(\alpha-\lambda)} = \frac{w}{r+\lambda} + \frac{r\lambda w_R - r\lambda b + \alpha\lambda w_R - \lambda^2 b - (r+\lambda)\alpha w_R + (r+\lambda)\lambda b}{r(\alpha-\lambda)(r+\lambda)}$$

$$= \frac{w}{r+\lambda} + \frac{r\lambda w_R - r\lambda b + \alpha\lambda w_R - \lambda^2 b - r\alpha w_R - \lambda\alpha w_R + r\lambda b + \lambda^2 b}{r(\alpha-\lambda)(r+\lambda)}$$

$$= \frac{w}{r+\lambda} + \frac{r\lambda w_R - r\alpha w_R}{r(\alpha-\lambda)(r+\lambda)} = \frac{w}{r+\lambda} - \frac{r(\alpha-\lambda)w_R}{r(\alpha-\lambda)(r+\lambda)}$$

$$\Rightarrow \quad W(w) - U = \frac{w-w_R}{r+\lambda}$$

The last step is to evaluate (19) when $w = w_R$ and equate that with (11):

$$w_R + \lambda \int_0^\infty max\{W(w') - U, U - U\}\, dF(w) = b + \alpha \int_{w_R}^\infty [W(w) - U] dF(w)$$

$$\Rightarrow \quad w_R = b + \alpha \int_{w_R}^\infty [W(w) - U] dF(w) - \lambda \int_{w_R}^\infty [W(w) - U] dF(w) = b + (\alpha - \lambda) \int_{w_R}^\infty [W(w) - U] dF(w)$$

Finally insert for $W(w) - U$ and get:

$$w_R = b + \frac{\alpha-\lambda}{r+\lambda} \int_{w_R}^\infty (w - w_R) dF(w) \tag{20}$$

Notice that $\alpha < \lambda$ implies $w_R < b$. In this case, workers accept a job paying less than unemployment income and wait for the wage to change, rather than searching while unemployed. In any case the usual comparative static results, such as $\partial w_R / \partial b > 0$, are similar to what we found earlier.

3.2 *Job-to-Job Transitions*

To explain how workers change employers without an intervening spell of unemployment, we need to consider on-the-job search. Suppose new offers arrive at a rate α_0 when unemployed and α_1 while employed. Each offer is an i.i.d. draw from F. Assume that workers also lose their job exogenously at a rate λ. The Bellman equations are:

$$rU = b + \alpha_0 \int_{w_R}^{\infty} [W(w) - U] dF(w) \tag{21}$$

$$rW(w) = w + \alpha_1 \int_0^{\infty} max\{W(w') - W(w), 0\} \, dF(w') + \lambda[U - W(w)] \tag{22}$$

Evaluating (22) when $W(w_R) = U$, we get:

$$rU = w_R + \alpha_1 \int_{w_R}^{\infty} [W(w') - U] dF(w') + \lambda[U - U] = w_R + \alpha_1 \int_{w_R}^{\infty} [W(w') - U] dF(w')$$

Set this equal to (21) and solve for w_R:

$$w_R + \alpha_1 \int_{w_R}^{\infty} [W(w') - U] dF(w') = b + \alpha_0 \int_{w_R}^{\infty} [W(w') - U] dF(w')$$

$$\Rightarrow \quad w_R = b + (\alpha_0 - \alpha_1) \int_{w_R}^{\infty} [W(w') - U] dF(w') \tag{23}$$

$w_R > b$ if and only if $\alpha_0 > \alpha_1$. Thus, if a worker gets more offers when employed than unemployed, he is willing to accept wages below b. To eliminate W from the integral, we need some additional steps. Using integration by part, we can write the integral in (23) as:

$$\int_{w_R}^{\infty} [W(w') - W(w_R)] dF(w') = \int_{w_R}^{\infty} W(w') f(w') dw - \int_{w_R}^{\infty} W(w_R) f(w') dw$$

$$= |W(w')F(w')|_{w_R}^{\infty} - \int_{w_R}^{\infty} W'(w')F(w')dw - |W(w_R)F(w')|_{w_R}^{\infty}$$

$$= [W(\infty)F(\infty) - W(w_R)F(w_R)] - \int_{w_R}^{\infty} W'(w')F(w')dw - [W(w_R)F(\infty) - W(w_R)F(w_R)]$$

$$= [W(\infty) - W(w_R)] - \int_{w_R}^{\infty} W'(w')F(w')dw = \int_{w_R}^{\infty} W'(w')dw - \int_{w_R}^{\infty} W'(w')F(w')dw$$

$$= \int_{w_R}^{\infty} W'(w')[1 - F(w')]dw$$

To find an alternative expression for $W'(w')$, we differentiate (22) with respect to w and solve for $W'(w)$:

$$rW'(w) = 1 - \alpha_1[1 - F(w)]W'(w) - \lambda W'(w')$$

$$\Rightarrow \quad \{r + \lambda + \alpha_1[1 - F(w)]\}W'(w) = 1$$

$$\Rightarrow \quad W'(w) = \frac{1}{r+\lambda+\alpha_1[1-F(w)]}$$

Substitute for $W'(w')$ into the integral:

$$\int_{w_R}^{\infty} \frac{1-F(w')}{r+\lambda+\alpha_1[1-F(w)]} dw$$

Thus, (23) can be written:

$$w_R = b + (\alpha_0 - \alpha_1) \int_{w_R}^{\infty} \frac{1-F(w)}{r+\lambda+\alpha_1[1-F(w)]} dw \tag{24}$$

If $\alpha_1 = 0$, this reduces to the earlier reservation wage equation (13). Many results, like $\frac{\partial w_R}{\partial b} > 0$, are similar to what we found above, but we also have some new predictions. For instance, when w_R is higher, workers are less likely to accept a low w, so they are less likely to experience job-to-job transitions. Thus, an increase in b reduces turnover. The model derived also makes predictions about wages, tenure, and separation rates. For example, workers typically move up the wage distribution during an employment spell, so the time since a worker was last unemployed is positively correlated with his wage. Also, workers who earn higher wages are less likely to get better job opportunities, generating a negative correlation between wages and separation rates. And the fact that a worker has held a job for a long time typically means he is unlikely to find a better one, generating a negative relationship between job tenure and separation rates. All of these features are consistent with the empirical evidence. With a slight reinterpretation, the framework can also be used to discuss aggregate variables. Suppose there are many workers, each solving a problem like the one discussed above, with the various stochastic events (like offer arrivals) i.i.d. across workers. Each unemployed worker becomes employed at rate $H = \alpha_0[1 - F(w_R)]$ and each employed worker looses his job at rate λ, so the aggregate unemployment rate u evolves according to:

$$\dot{u} = \lambda(1-u) - \alpha_0[1 - F(w_R)]u$$

Over time, this converges to the steady state:

$$\{\lambda + \alpha_0[1 - F(w_R)]\}u = \lambda$$

$$\Rightarrow \quad u^* = \frac{\lambda}{\lambda+\alpha_0[1-F(w_R)]} \tag{25}$$

One can also calculate the cross-sectional distribution of observed wages for employed workers, denoted by $G(w)$, given any offer distribution $F(w)$. For all $w \geq w_R$, the flow of workers into employment at a wage no greater than w is:

$$u\alpha_0[F(w) - F(w_R)]$$

or the number of unemployed workers times the rate at which they find a job paying between w_R and w. The flow of workers out of this state is:

$$(1-u)G(w)\{\lambda+\alpha_1[1-F(w)]\}$$

or the number of workers employed at w or less, times the rate at which they leave either for exogenous reasons or because they get an offer above w. In steady state, these flows are equal. Using (25) and rearranging, we therefore get:

$$(1-u)G(w)\{\lambda+\alpha_1[1-F(w)]\}=u\alpha_0[F(w)-F(w_R)]$$

$$\Rightarrow\quad \left(1-\frac{\lambda}{\lambda+\alpha_0[1-F(w_R)]}\right)G(w)\{\lambda+\alpha_1[1-F(w)]\}=\frac{\lambda}{\lambda+\alpha_0[1-F(w_R)]}\alpha_0[F(w)-F(w_R)]$$

$$\Rightarrow\quad \frac{\alpha_0[1-F(w_R)]}{\lambda+\alpha_0[1-F(w_R)]}G(w)\{\lambda+\alpha_1[1-F(w)]\}=\frac{\lambda}{\lambda+\alpha_0[1-F(w_R)]}\alpha_0[F(w)-F(w_R)]$$

$$\Rightarrow\quad G(w)=\frac{\lambda[F(w)-F(w_R)]}{[1-F(w_R)]\{\lambda+\alpha_1[1-F(w)]\}} \tag{26}$$

We can now compute the steady state job-to-job transition rate, given by:

$$\alpha_1\int_{w_R}^{\infty}[1-F(w)]dG(w) \tag{27}$$

4. *Random Matching and Bargaining*

4.1 Matching

We will now introduce a popular line of research where meetings between workers and firms are determined through a matching function, and wages are determined through bargaining. Suppose that at some point in time there are v vacancies posted by firms looking for workers and u unemployed workers looking for jobs. Each firm has one job, i.e. it hires at most one worker. The flow of contacts between firms and workers is given by a matching technology m:

$$m=m(u,v)$$

m is interpreted as the rate at witch matches occur. Assume that the function m is continuous, nonnegative, increasing in both arguments and concave, with $m(0,v)=m(u,0)=0$. It is also convenient to assume constant returns to scale, i.e. $\chi m=m(\chi u,\chi v)$. The most popular specification of the matching function is Cobb-Douglas, i.e.:

$$m=Au^{\alpha}v^{1-\alpha}$$

Under the assumption that all workers are the same and all firms are the same, the arrival rates for unemployed workers and employers with vacancies are given by:

$$\alpha_w=\frac{m(u,v)}{u},\qquad \alpha_e=\frac{m(u,v)}{v} \tag{28}$$

When m is Cobb-Douglas, these two become:

$$\alpha_w = \frac{Au^\alpha v^{1-\alpha}}{u} = A\left(\frac{v}{u}\right)^{1-\alpha}, \qquad \alpha_e = \frac{Au^\alpha v^{1-\alpha}}{v} = A\left(\frac{u}{v}\right)^{\alpha}$$

More generally, given constant returns to scale in m, α_w and α_e depend only on the ratio $\frac{v}{u}$, referred to as a measure of market tightness. α_w is an increasing and α_e a decreasing function of the market tightness in this case, and there is a 1 to 1 relationship between α_w and α_e.

4.2 Bargaining

Consider the situation of a worker and a firm who have met and have an opportunity to produce a flow of output y. Suppose that if the worker gets a wage w, his lifetime utility is $W(w)$ while the firm earns expected discounted profit $J(\pi)$, where:

$$\pi = y - w$$

If the two parties fail to reach an agreement, the worker's payoff falls to U and the firm's to V. For the moment we take these two values for given. Assume that w is determined by the generalized Nash bargaining solution with threat points U and V:

$$w \in arg\,max\left\{[W(w) - U]^\theta [J(y-w) - V]^{1-\theta}\right\} \tag{29}$$

where $\theta \in (0,1)$ is the workers relative bargaining power, and the problem is subject to the constraints $W(w) - U \geq 0$ and $J(\pi) - V \geq 0$. The solution to (29) is found from the first order condition:

$$\theta[W(w) - U]^{\theta-1} W'(w)[J(y-w) - V]^{1-\theta} - (1-\theta)[W(w) - U]^\theta [J(y-w) - V]^{-\theta} J'(y-w) = 0$$

$$\Rightarrow \quad \theta[J(y-w) - V]W'(w) = (1-\theta)[W(w) - U]J'(y-w) \tag{30}$$

(30) can be solved for w. To proceed, suppose as usual that workers and firms are risk neutral, infinitely lived, and discount future payoffs in continuous time at rate r, and that matches end exogenously at rate λ. Then we have:

$$rW(w) = w + \lambda[U - W(w)] \tag{31}$$

$$rJ(\pi) = \pi + \lambda[V - J(\pi)] \tag{32}$$

From (31):

$$rW'(w) = 1 - \lambda W'(w)$$

$$\Rightarrow \quad W'(w) = \frac{1}{r+\lambda}$$

From (32):

$$-rJ'(\pi) = -1 + \lambda J'(\pi)$$
$$\Rightarrow \quad J'(\pi) = \frac{1}{r+\lambda}$$

Inserting these two into (30) and rearranging gives:

$$\frac{\theta[J(\pi)-V]}{r+\lambda} = \frac{(1-\theta)[W(w)-U]}{r+\lambda}$$
$$\Rightarrow \quad W(w) - U = \theta[J(\pi) - V + W(w) - U]$$
$$\Rightarrow \quad W(w) = U + \theta[J(\pi) - V + W(w) - U] \tag{33}$$

Thus, in terms of total lifetime utility the worker receives his threat point U plus a share θ of the total surplus, denoted S and defined by:

$$S = J(\pi) - V + W(w) - U$$

An alternative expression for the surplus is found by solving (31) and (32) for $W(w)$ and $J(\pi)$, respectively. From (31):

$$(r+\lambda)W(w) = w + \lambda U$$
$$\Rightarrow \quad W(w) = \frac{w+\lambda U}{r+\lambda}$$

From (32):

$$(r+\lambda)J(\pi) = \pi + \lambda V$$
$$\Rightarrow \quad J(\pi) = \frac{\pi+\lambda V}{r+\lambda}$$

The total surplus now becomes:

$$S = \frac{y-w+\lambda V-(r+\lambda)V}{r+\lambda} + \frac{w+\lambda U-(r+\lambda)U}{r+\lambda} = \frac{y-rU-rV}{r+\lambda} \tag{34}$$

Notice that S does not depend on the wage. What about reservation wages and profits? By evaluating (31) when $W(w_R) = U$, we get:

$$rW(w_R) = w_R + \lambda[U - W(w_R)]$$
$$\Rightarrow \quad rU = w_R$$

Thus:

$$W(w) - U = \frac{w+\lambda U-(r+\lambda)U}{r+\lambda} = \frac{w-rU}{r+\lambda} = \frac{w-w_R}{r+\lambda}$$

Evaluating (32) when $J(\pi_R) = V$ we get:

$$rJ(\pi_R) = \pi_R + \lambda[V - J(\pi_R)]$$

$\Rightarrow \quad rV = \pi_R$

Thus:

$J(\pi) - V = \frac{\pi + \lambda V - (r+\lambda)V}{r+\lambda} = \frac{\pi - rV}{r+\lambda} = \frac{\pi - \pi_R}{r+\lambda}$

Then (29) reduces to:

$$w \in arg\,max\{(w - w_R)^{\theta}(\pi - \pi_R)^{1-\theta}\} \tag{35}$$

and the solution follows as:

$\theta(w - w_R)^{\theta-1}(\pi - \pi_R)^{1-\theta} - (1-\theta)(w - w_R)^{\theta}(\pi - \pi_R)^{-\theta} = 0$

$\Rightarrow \quad \theta \frac{\pi - \pi_R}{w - w_R} = 1 - \theta$

$\Rightarrow \quad \theta(y - w - \pi_R) = (1-\theta)(w - w_R)$

$$\Rightarrow \quad w = w_R + \theta(y - \pi_R - w_R) \tag{36}$$

Hence, in this model the Nash solution also splits the surplus in terms of the current period utility. Notice that $w \geq w_R$ if and only if $y \geq y_R = \pi_R + w_R$. Similarly, $\pi = y - w \geq \pi_R$ if and only if $y \geq y_R$. Hence, workers and firms agree to consummate relationships if and only if $y \geq y_R$.

4.3 Equilibrium

We now combine matching and bargaining in a model where a firm's decision to post a vacancy is endogenized using a free entry condition. There is a unit mass of homogenous workers, and unmatched workers search costlessly while matched workers cannot search. Remember that α_w is the flow rate into employment from the pool of unemployed, while λ is the job destruction rate. Thus, the evolution of mean unemployment follows:

$\dot{u} = \lambda(1-u) - \alpha_w u$

The steady state unemployment rate u^* is constant, i.e. given by $\dot{u} = 0$:

$(\lambda + \alpha_w)u^* = \lambda$

$\Rightarrow \quad u^* = \frac{\lambda}{\lambda + \alpha_w}$

α_w depends positively on v, so there is a downward sloping, convex relationship between u and v, typically referred to as the Beveridge curve. Assuming constant returns, we know that once α_w is determined, so is α_e. This is because both are given by the market tightness. The value of posting a vacancy is:

$$rV = -k + \alpha_e[J(\pi) - V] \tag{37}$$

where k is a flow cost, e.g. recruitment costs, of posting a vacancy. Free entry drives $V = 0$ and (37) yields:

$$\alpha_e J(\pi) = k \tag{38}$$

The value of unemployment satisfies:

$$rU = b + \alpha_w [W(w) - U] \tag{39}$$

while the equations for $W(w)$ and $J(\pi)$ are unchanged from (31) and (32). Formally, an equilibrium includes the value functions (J, W, U), the wage w, and the unemployment and vacancy rates (u, v), satisfying the Bellman equations, the bargaining solution, free entry, and the steady state condition. (34) and free entry gives:

$$(r + \lambda)S = y - rU \tag{40}$$

(33) allows us to write (39) as:

$$rU = b + \alpha_w \theta S$$

Using this, (40) becomes:

$$(r + \lambda)S = y - b - \alpha_w \theta S$$
$$\Rightarrow \quad (r + \lambda + \alpha_w \theta)S = y - b \tag{41}$$

Furthermore, (30) implies:

$$\theta J(\pi) = (1 - \theta)[W(w) - U] = (1 - \theta)\theta S$$
$$\Rightarrow \quad J(\pi) = (1 - \theta)S$$

Thus, (38) is:

$$k = \alpha_e (1 - \theta)S \tag{42}$$

Equilibrium is completely characterized by (41) and (42). Combining them we get:

$$(r + \lambda + \alpha_w \theta) \frac{k}{\alpha_e (1 - \theta)} = y - b$$
$$\Rightarrow \quad \frac{r + \lambda + \alpha_w \theta}{\alpha_e (1 - \theta)} = \frac{y - b}{k} \tag{43}$$

From (31) and the free entry condition we have:

$$J(\pi) - V = \frac{\pi + \lambda V}{r + \lambda} - V$$
$$\Rightarrow \quad J(\pi) = \frac{\pi}{r + \lambda}$$

Combining that with $J(\pi) = (1-\theta)S$ the wage is found:

$$\frac{y-w}{r+\lambda} = (1-\theta)S$$
$$\Rightarrow \quad w = y - (r+\lambda)(1-\theta)S \tag{44}$$

Suppose b rises. Then α_w falls while α_e increases. The latter reduces S because of (42), and raises w because of (43). Finally, the average unemployment duration $D = \frac{1}{H} = \frac{1}{\alpha_w[1-F(w_R)]}$ becomes longer. This is because of α_w declines while at the same time $F(w_R)$ increases.

4.4 Match-Specific Productivity

In the above model, it takes time for workers and firms to get together, but every contact leads to a match and w is the same in every match. Now we extend the model so that not every contact leads to a match and not every match has the same w. Assume that when a worker and a firm meet they draw match-specific productivity y from a distribution F, where y is observed by both and constant for the duration of the match. From section 4.2, workers and firms agree to match if and only if $y \geq y_R$. In equilibrium workers in a match with productivity y earn $w(y)$ satisfying the Nash bargaining solution. Let $W_y(w)$ be the value for an employed worker of a match with productivity y earning $w, J_y(y-w)$ the value for a firm with a filled job at productivity y earning profits $y-w$, and S_y the surplus in a job with productivity y. Generalizing (40) gives:

$$(r+\lambda)S_y = y - rU \tag{45}$$

Only the Bellman equations for an unemployed worker and the free entry condition change appreciably, becoming:

$$rU = b + \alpha_w \int_{y_R}^{\infty} \{W[w(y)] - U\} dF(y) = b + \alpha_w \theta \int_{y_R}^{\infty} S_y dF(y) \tag{46}$$
$$k = \alpha_e \int_{y_R}^{\infty} J_y[y - w(y)] dF(y) = \alpha_e (1-\theta) \int_{y_R}^{\infty} S_y dF(y) \tag{47}$$

where (47) follows from (38). Insert (47) into (46) and get:

$$rU = b + \frac{\alpha_w \theta k}{\alpha_e(1-\theta)}$$

Substitute that back into (45):

$$(r+\lambda)S_y = y - b - \frac{\alpha_w \theta k}{\alpha_e(1-\theta)} \tag{48}$$

Evaluating (34) when $y = y_R$, i.e. when $w = w_R$ we get:

$$S_{y_R} = J(y_R - w_R) - V + W(w_R) - U = \frac{\pi_R - \pi_R}{r+\lambda} + \frac{w_R - w_R}{r+\lambda} = 0$$

Thus, when $y = y_R$ (48) becomes:

$$0 = y_R - b - \frac{\alpha_w \theta k}{\alpha_e(1-\theta)}$$

$$\Rightarrow \quad y_R = b + \frac{\alpha_w \theta k}{\alpha_e(1-\theta)} \tag{49}$$

Substituting (49) into (48) we get:

$$S_y = \frac{y - b - \frac{\alpha_w \theta k}{\alpha_e(1-\theta)}}{r+\lambda} = \frac{y - y_R}{r+\lambda}$$

Finally, when that result is inserted into (47):

$$k = \alpha_e(1-\theta) \int_{y_R}^{\infty} \frac{y - y_R}{r+\lambda} dF(y)$$

$$\Rightarrow \quad (r+\lambda)k = \alpha_e(1-\theta) \int_{y_R}^{\infty} (y - y_R) dF(y) \tag{50}$$

We can now solve for y_R and α_w from (49) and (50). The first of these equations describes an increasing relationship between α_w and y_R. When it is easier to find a job, the worker is more willing to turn down a potential match with low productivity. The second gives a decreasing relationship between y_R and α_w. When y_R is higher, matches are less profitable for firms so they post fewer vacancies. There exists a unique equilibrium under standard conditions. One can again recover the wage function. Inserting for S_y into (44):

$$w(y) = y - (r+\lambda)(1-\theta)S_y = y - (r+\lambda)(1-\theta)\frac{y - y_R}{r+\lambda} = y - (1-\theta)(y - y_R)$$

$$\Rightarrow \quad w(y) = y_R + \theta(y - y_R)$$

Suppose there is an increase in b. Then (49) shifts, but not (50), resulting in an increase in y_R, a reduction in α_w, and a reduction in $H = \alpha_w[1 - F(y_R)]$.

4.5 Endogenous Separations

One can also endogenize the separation rate by incorporating on-the-job wage changes. The resulting framework captures endogenously both the flows into and out of employment. Let y be current productivity in a match, and assume that at a rate λ we get a new draw from $F(y'|y)$, where $F(y'|y_2)$ first order stochastically dominates $F(y'|y_1)$ whenever $y_2 > y_1$. Assume all matches start with the same y_0. An equilibrium is defined as a natural extension of the previous model, and we can jump directly to the equation for the surplus:

$$(r+\lambda)S_y = y - rU + \lambda \int_{w_R}^{\bar{y}} S_{y'} dF(y'|y) \tag{51}$$

Since $rU = b + \alpha_w \theta S_{y_0}$ we get:

$$(r+\lambda)S_y = y - b - \alpha_w \theta S_{y_0} + \lambda \int_{w_R}^{\bar{y}} S_{y'} dF(y'|y) \tag{52}$$

To close the model we again use free entry:

$$k = \alpha_e(1-\theta)S_{y_0} \tag{53}$$

Finding equilibrium amounts to solving (53) and (52) for y_R and α_w. This is a fixed point problem in a system of functional equations which is more complicated to solve – (52) defines both y_R and S_y as functions of α_w. Nevertheless, an increase in α_w reduces S_y for all y and hence raises the reservation wage y_R. Thus, (52) describes an increasing relationship between α_w and y_R. At the same time, (53) indicates that when α_w is higher S_{y_0} must be higher, so from (52) y_R must be lower, and this defines a decreasing relationship between α_w and y_R. The intersection of these curves gives steady-state equilibrium, which exists uniquely under standard conditions.

5. Directed Search and Posting

5.1 A One-Shot Model

We now move to models where some agents can post offers, and other agents direct their search to the most attractive alternatives. The combination of posting and directed search is known as competitive search. Suppose firms post a wage, and then workers can choose where to apply. At the beginning of the period, there are large numbers u and v of unemployed workers and vacancies, and:

$$q = \frac{u}{v}$$

is the queue length. Each firm with a vacancy must pay a fixed cost k. We can assume either free entry, making v endogenous, or fix the number of vacancies. Any match within the period produces output y, which is divided between the worker and the firm according to a posted wage. At the end of the period, unmatched workers get b, while unmatched vacancies get 0. Then the model ends. The idea of competitive search is that the firm has no market power and takes the market value of a worker as given. Consider a worker facing a menu of different wages. Let U denote the highest value that he can get by applying for a job at some firm. Then a worker is willing to apply to a particular job offering a wage $w \geq b$ only if he believes the queue length q at that job is sufficiently small. In fact, he is willing to apply only if $\alpha_w(q)$ is sufficiently large, in the sense that:

$$U \leq \alpha_w(q)w + [1-\alpha_w(q)]b \tag{54}$$

If (54) is not met, no worker will show up, since all workers will go to other firms. If the inequality is strict, all workers would apply to this firm, reducing the right hand side. Therefore, in equilibrium, if any workers apply to a particular job, q adjusts to satisfy (54) with equality. To an employer, (54) describes how a change in the wage affects his queue length. Therefore he chooses w to maximize:

$$V = max_{w,q}\{-k + \alpha_e(q)(y - w)\} \quad (55)$$

subject to (54). Notice that the definitions of the probabilities of a job offer for the worker:

$$\alpha_w(q) = \frac{m(u,v)}{u}$$

and a worker applying to a firm

$$\alpha_e(q) = \frac{m(u,v)}{v}$$

lead to the relation:

$$\frac{\alpha_e(q)}{\alpha_w(q)} = \frac{\frac{m(u,v)}{v}}{\frac{m(u,v)}{u}} = \frac{u}{v} = q$$

$$\Rightarrow \quad \alpha_e(q) = \alpha_w(q)q$$

Thus, (54) can be rewritten:

$$U - b = \alpha_w(q)(w - b)$$

$$\Rightarrow \quad \frac{U-b}{\alpha_w(q)} + b = w$$

$$\Rightarrow \quad w = q\frac{U-b}{\alpha_e(q)} + b$$

Insert into (55) and get:

$$V = max_{w,q}\left\{-k + \alpha_e(q)\left(y - q\frac{U-b}{\alpha_e(q)} - b\right)\right\}$$

$$\Rightarrow \quad V = max_q\{-k + \alpha_e(q)(y - b) - q(U - b)\} \quad (56)$$

The first order condition $\frac{\partial V}{\partial q} = 0$ yields:

$$\alpha_e'(q)(y - b) - (U - b) = 0$$

$$\Rightarrow \quad \alpha_e'(q)(y - b) = U - b \quad (57)$$

In particular, (57) implies all employers choose the same q, which in equilibrium must equal the economywide q^*. Thus, (57) for pins down the equilibrium value of U:

$$U = \alpha_e'(q^*)(y - b) + b$$

To get the equilibrium wage, we substitute for U into (54) and solve for w^*:

$$\alpha_e'(q^*)(y-b)+b=\alpha_w(q^*)w^*+[1-\alpha_w(q^*)]b$$
$$\Rightarrow \quad \alpha_e'(q^*)(y-b)+\alpha_w(q^*)b=\alpha_w(q^*)w^*$$
$$\Rightarrow \quad w^*=b+\frac{\alpha_e'(q^*)(y-b)}{\alpha_w(q^*)}=b+\frac{\alpha_e'(q^*)(y-b)}{\frac{\alpha_e(q^*)}{q^*}}=b+\frac{q^*\alpha_e'(q^*)}{\alpha_e(q^*)}(y-b)$$
$$\Rightarrow \quad w^*=b+\varepsilon(q^*)(y-b) \tag{58}$$

Note that the equilibrium wage is strictly above b. Substituting (58) into (55) pins down V:

$$V=-k+\alpha_e(q^*)\left[y-b-\frac{q^*\alpha_e'(q^*)}{\alpha_e(q^*)}(y-b)\right]$$
$$\Rightarrow \quad V=-k+[\alpha_e(q^*)-q^*\alpha_e'(q^*)](y-b) \tag{59}$$

If the number of vacancies v is fixed, this gives profit. Alternatively we can use free entry to endogenize v and hence q^*. With free entry firms enter until profits are non-positive:

$$V\leq 0$$

In equilibrium we must have $V=0$. Thus, we can determine q^* implicitly from this zero-profit condition. Notice that this model looks a lot like a one-shot version of the random search and bargaining model. There is a key difference, however. Here the surplus share is endogenously determined.

5.2 Directed Matching

TBA

5.3 A Dynamic Model

To get something like the basic Pissarides model with directed search, consider an unemployed worker who anticipates an unemployment-vacancy ratio q and a wage w. Then:

$$rU=b+\alpha_w(q)[W(w)-U] \tag{60}$$
$$rW(w)=w+\lambda[U-W(w)] \tag{61}$$

Solving (61) for $W(w)$ we get:

$$(r+\lambda)W(w)=w+\lambda U$$
$$\Rightarrow \quad W(w)=\frac{w+\lambda U}{r+\lambda}$$

Now (61) and (60) can be combined:

$$rU=b+\alpha_w(q)\left[\frac{w+\lambda U}{r+\lambda}-U\right]=b+\alpha_w(q)\frac{w+\lambda U-(r+\lambda)U}{r+\lambda}$$
$$\Rightarrow \quad rU=b+\alpha_w(q)\frac{w-rU}{r+\lambda} \tag{62}$$

For firms, the value when the job is unfilled is:

$$rV = -k + \alpha_e(q)[J(y-w) - V] \tag{63}$$

while the value of a filled job becomes:

$$rJ(y-w) = y - w + \lambda[V - J(y-w)] \tag{64}$$

Notice that (61) and (64) are the same as (31) and (32). To proceed we solve (64) for $J(y-w)$:

$$(r+\lambda)J(y-w) = y - w + \lambda V$$
$$\Rightarrow \quad J(y-w) = \frac{y-w+\lambda V}{r+\lambda}$$

Insert the result into (63):

$$rV = -k + \alpha_e(q)\left(\frac{y-w+\lambda V}{r+\lambda} - V\right) = -k + \alpha_e(q)\frac{y-w+\lambda V-(r+\lambda)V}{r+\lambda}$$
$$\Rightarrow \quad rV = -k + \alpha_e(q)\frac{y-w-rV}{r+\lambda}$$

Using the free entry condition $V = 0$ we get:

$$k = \frac{\alpha_e(q)(y-w)}{r+\lambda} \tag{65}$$

Thus, firms maximize (65) subject to (62) with respect to q and w. We can solve the constraint for w:

$$rU + \frac{(r+\lambda)(rU-b)}{\alpha_w(q)} = w$$

and substitute for w in (65):

$$k = \frac{\alpha_e(q)\left[y-rU-\frac{(r+\lambda)(rU-b)}{\alpha_w(q)}\right]}{r+\lambda} = \alpha_e(q)\frac{y-rU}{r+\lambda} - \frac{\alpha_e(q)}{\alpha_w(q)}(rU-b) = \alpha_e(q)\frac{y-rU}{r+\lambda} - q(rU-b)$$

Thus, firms face the following maximization problem:

$$max_q\left\{\alpha_e(q)\frac{y-rU}{r+\lambda} - q(rU-b)\right\} \tag{66}$$

The first order condition yields:

$$\alpha_e'(q)\frac{y-rU}{r+\lambda} - (rU-b) = 0$$
$$\Rightarrow \quad \alpha_e'(q)\frac{y-rU}{r+\lambda} = rU - b \tag{67}$$

Equation (67) has a unique solution, so all firms choose the same q. The last step is to get an implicit expression for q. First we solve (67) for rU:

$$rU + \alpha_e'(q)\frac{rU}{r+\lambda} = b + \frac{\alpha_e'(q)y}{r+\lambda}$$

$$\Rightarrow \quad \frac{r+\lambda+\alpha_e'(q)}{r+\lambda} rU = b + \frac{\alpha_e'(q)y}{r+\lambda}$$

$$\Rightarrow \quad rU = \frac{r+\lambda}{r+\lambda+\alpha_e'(q)} b + \frac{\alpha_e'(q)y}{r+\lambda+\alpha_e'(q)}$$

Second we solve (62) for rU:

$$rU + \alpha_w(q)\frac{rU}{r+\lambda} = b + \frac{\alpha_w(q)w}{r+\lambda}$$

$$\Rightarrow \quad \frac{r+\lambda+\alpha_w(q)}{r+\lambda} rU = b + \frac{\alpha_w(q)w}{r+\lambda}$$

$$\Rightarrow \quad rU = \frac{r+\lambda}{r+\lambda+\alpha_w(q)} b + \frac{\alpha_w(q)w}{r+\lambda+\alpha_w(q)}$$

Third, we equate the two and solve for $\alpha_e(q)w$ using $\alpha_e(q) = \alpha_w(q)q$:

$$\frac{r+\lambda}{r+\lambda+\alpha_e'(q)} b + \frac{\alpha_e'(q)y}{r+\lambda+\alpha_e'(q)} = \frac{r+\lambda}{r+\lambda+\alpha_w(q)} b + \frac{\alpha_w(q)w}{r+\lambda+\alpha_w(q)}$$

$$\Rightarrow \quad \frac{\alpha_w(q)w}{r+\lambda+\alpha_w(q)} = \frac{r+\lambda}{r+\lambda+\alpha_e'(q)} b + \frac{\alpha_e'(q)y}{r+\lambda+\alpha_e'(q)} - \frac{r+\lambda}{r+\lambda+\alpha_w(q)} b$$

$$\Rightarrow \quad \frac{\alpha_e(q)}{q} w = \frac{(r+\lambda)b+\alpha_e'(q)y}{r+\lambda+\alpha_e'(q)} \left[r+\lambda+\frac{\alpha_e(q)}{q}\right] - (r+\lambda)b$$

$$\Rightarrow \quad \alpha_e(q)w = \frac{(r+\lambda)b+\alpha_e'(q)y}{r+\lambda+\alpha_e'(q)} [(r+\lambda)q + \alpha_e(q)] - q(r+\lambda)b$$

Fourth, we substitute the result into the free entry condition (65) and rewrite:

$$(r+\lambda)k = \alpha_e(q)y - \alpha_e(q)w = \alpha_e(q)y - \frac{(r+\lambda)b + \alpha_e'(q)y}{r+\lambda+\alpha_e'(q)} [(r+\lambda)q + \alpha_e(q)] + q(r+\lambda)b$$

$$= \frac{[r+\lambda+\alpha_e'(q)]\alpha_e(q)y - [(r+\lambda)b + \alpha_e'(q)y][(r+\lambda)q + \alpha_e(q)] + [r+\lambda+\alpha_e'(q)]q(r+\lambda)b}{r+\lambda+\alpha_e'(q)}$$

Multiply the denominator over to the left hand side and simplify:

$$\begin{aligned}[r+\lambda+\alpha_e'(q)](r+\lambda)k \\ &= (r+\lambda)\alpha_e(q)y + \alpha_e'(q)\alpha_e(q)y - (r+\lambda)^2bq - (r+\lambda)b\alpha_e(q) - \alpha_e'(q)y(r+\lambda)q \\ &- \alpha_e'(q)y\alpha_e(q) + (r+\lambda)^2bq + \alpha_e'(q)(r+\lambda)bq\end{aligned}$$

$$\Rightarrow \quad [r+\lambda+\alpha_e'(q)](r+\lambda)k = (r+\lambda)\alpha_e(q)y - (r+\lambda)b\alpha_e(q) - \alpha_e'(q)y(r+\lambda)q + \alpha_e'(q)(r+\lambda)bq$$

$$\Rightarrow \quad [r+\lambda+\alpha_e'(q)](r+\lambda)k = (r+\lambda)y[\alpha_e(q) - \alpha_e'(q)q] - (r+\lambda)b[\alpha_e(q) - \alpha_e'(q)]$$

$$\Rightarrow \quad [r+\lambda+\alpha_e'(q)](r+\lambda)k = (r+\lambda)[\alpha_e(q) - \alpha_e'(q)q](y-b)$$

$$\Rightarrow \quad [r+\lambda+\alpha_e'(q)]k = [\alpha_e(q) - \alpha_e'(q)q](y-b)$$

$$\Rightarrow \quad \frac{r+\lambda+\alpha_e'(q)}{\alpha_e(q)-\alpha_e'(q)q} = \frac{y-b}{k} \tag{68}$$

This pins down the equilibrium q, or equivalently the arrival rates α_w and α_e. As before a raise in b raises q, reduces α_w, raises α_e, and increases w. Although the effects are similar as in the previous section, the mechanism is different. An increase in b in this model makes workers more willing to accept an increase in the risk of unemployment in return for an increase in w. Firms respond by offering workers what they want, fewer jobs at higher wages.

6. *Random Matching and Posting*

6.1 Heterogeneous Leisure

We know combine random matching, as in section 4, with posting, as in section 5. This class of models is popular in the literature on wage dispersion, which tries to understand how workers with identical productivity can be paid different wages. Diamond (1971) developed a model with identical workers and random matching, an early attempt to construct a model of dispersion. However, in that model there is a unique equilibrium where all employers set the same wage, $w = b$. The reason is that all workers choose the same w_R since they are homogenous. What about firms? Clearly, no firm set $w < w_R$ as this would mean no hiring, and no firm posts $w > w_R$ as it can hire any worker it contacts at w_R. Furthermore, suppose all firms post $w > b$. If one firm deviates and offer a slightly lower wage it still hires every worker it meets as long as $w > b$. Thus, this model not only fails to rationalize wage dispersion, it fails to explain why workers are searching in the first place. One solution to this puzzle is based on the observation that search frictions produce a natural trade-off for the firm: while higher wages means lower profit per worker, it allows for hiring workers faster, and so, in the long run, you get more of them. In Diamonds model there is no increase in hiring rates when wages are increased. One way to generate dispersion is to allow for heterogeneity in worker's value of leisure. Consider to types of workers, some with $b = b_1$ and other with $b = b_2 > b_1$. For any wage distribution F there are two reservation wages, w_1 and $w_2 > w_1$. If $W_i(w)$ is the value of a type i worker who is employed at wage w and U_i is that worker's unemployment value, these wages satisfy $W_1(w) = U_1$ and $W_2(w) = U_2$. Generalizing the logic from the Diamond model, no firm posts a wage other than w_1 or w_2. It is possible that these two wages yield equal profit however, since low-wage firms can hire only workers with $b = b_1$ while high-wage firms can hire everyone they contact. To see how this works, normalize the measure of firms to 1, and let the measure of workers be $L = L_1 + L_2$. Let σ be the fraction of firms posting w_2. Any candidate equilibrium wage dispersion is completely characterized by w_1, w_2 and σ. Observe first that the reservation wage of type 2 workers is $w_2 = b_2$. To determine w_1, note that type 1 workers accept both $w = w_1$ and $w = w_2$. Given the arrival rate α_w their value functions satisfy:

$$rU_1 = b_1 + \alpha_w(1-\sigma)[W_1(w_1) - U_1] + \alpha_w \sigma[W_1(w_2) - U_1] \tag{69}$$

$$rW_1(w_1) = w_1 + \lambda[U_1 - W_1(w_1)] \tag{70}$$

$$rW_1(w_2) = w_2 + \lambda[U_1 - W_1(w_2)] \tag{71}$$

Note that although type 1 workers accept w_1, they get no capital gain from doing so and suffer no capital loss when laid off, since $W_1(w_1) = U_1$. Then, using $w_2 = b_2$, we can simplify and solve for w_1 in terms of σ. From (70):

$$rU_1 = w_1$$

From (71):

$$(r+\lambda)W_1(w_2) = b_2 + \lambda U_1$$
$$\Rightarrow \quad W_1(w_2) = \frac{b_2+\lambda U_1}{r+\lambda}$$

From (69):

$$(r+\alpha_w\sigma)U_1 = b_1 + \alpha_w\sigma W_1(w_2)$$
$$\Rightarrow \quad W_1(w_2) = \frac{(r+\alpha_w\sigma)U_1 - b_1}{\alpha_w\sigma}$$

Equate the last two equations, rewrite, and then insert the first equation:

$$\frac{b_2+\lambda U_1}{r+\lambda} = \frac{(r+\alpha_w\sigma)U_1 - b_1}{\alpha_w\sigma}$$
$$\Rightarrow \quad \alpha_w\sigma b_2 + \alpha_w\sigma\lambda U_1 = (r+\lambda)(r+\alpha_w\sigma)U_1 - (r+\lambda)b_1$$
$$\Rightarrow \quad [(r+\lambda)(r+\alpha_w\sigma) - \alpha_w\sigma\lambda]U_1 = (r+\lambda)b_1 + \alpha_w\sigma b_2$$
$$\Rightarrow \quad r(r+\lambda+\alpha_w\sigma)U_1 = (r+\lambda)b_1 + \alpha_w\sigma b_2$$
$$\Rightarrow \quad w_1 = \frac{(r+\lambda)b_1 + \alpha_w\sigma b_2}{r+\lambda+\alpha_w\sigma} \qquad (72)$$

Unemployment among the two labor types evolves according to:

$$\dot{u}_1 = \lambda(1-u_1) - \alpha_w\sigma u_1 - \alpha_w(1-\sigma)u_1 = \lambda(1-u_1) - \alpha_w u_1$$
$$\dot{u}_2 = \lambda(1-u_2) - \alpha_w\sigma u_2$$

Thus, steady state employment is given by:

$$u_1 = \frac{\lambda}{\alpha_w+\lambda}$$
$$u_2 = \frac{\lambda}{\alpha_w\sigma+\lambda}$$

Thus, workers with a high leisure value are subject to more frequent unemployment. For firms, the expected value of contacting a worker is the probability that he accepts times the profit conditional on acceptance. The acceptance probability is $\frac{L_1u_1}{L_1u_1+L_2u_2}$ for firms paying w_1 and 1 for firms paying w_2, while discounted profit is $\frac{y-w_i}{r+\lambda}$. So expected discounted profits from the two wages are:

$$\Pi_1 = \alpha_e \frac{L_1 u_1}{L_1 u_1 + L_2 u_2} \frac{y - w_1}{r + \lambda} \tag{73}$$

$$\Pi_2 = \alpha_e \frac{y - b_2}{r + \lambda} \tag{74}$$

where for now we are taking α_e as given. Inserting for u_1, u_2 and w_1, algebra implies:

$$\Pi_2 - \Pi_1 = \frac{\alpha_e}{r+\lambda}\left[(y - b_2) - \frac{L_1 u_1}{L_1 u_1 + L_2 u_2}(y - w_1)\right]$$

$$= \frac{\alpha_e}{r+\lambda}\left\{(y - b_2) - \frac{L_1 \frac{\lambda}{\alpha_w + \lambda}}{L_1 \frac{\lambda}{\alpha_w + \lambda} + L_2 \frac{\lambda}{\alpha_w \sigma + \lambda}}\left[y - \frac{(r+\lambda) b_1 + \alpha_w \sigma b_2}{r + \lambda + \alpha_w \sigma}\right]\right\}$$

$$= \frac{\alpha_e}{r+\lambda}\left\{(y - b_2) - \frac{(\alpha_w \sigma + \lambda) L_1}{(\alpha_w \sigma + \lambda) L_1 + (\alpha_w + \lambda) L_2}\left[y - \frac{(r+\lambda) b_1 + \alpha_w \sigma b_2}{r + \lambda + \alpha_w \sigma}\right]\right\}$$

Multiplying both sides with $[(\alpha_w \sigma + \lambda) L_1 + (\alpha_w + \lambda) L_2](r + \lambda + \alpha_w \sigma)\frac{r+\lambda}{\alpha_e}$ we get:

$$[(\alpha_w \sigma + \lambda) L_1 + (\alpha_w + \lambda) L_2](r + \lambda + \alpha_w \sigma)\frac{r+\lambda}{\alpha_e}(\Pi_2 - \Pi_1)$$

$$= (r + \lambda + \alpha_w \sigma)(y - b_2)[(\alpha_w \sigma + \lambda) L_1 + (\alpha_w + \lambda) L_2] - (\alpha_w \sigma + \lambda) L_1 y (r + \lambda + \alpha_w \sigma)$$
$$+ (\alpha_w \sigma + \lambda) L_1 [(r+\lambda) b_1 + \alpha_w \sigma b_2]$$
$$= (r + \lambda + \alpha_w \sigma)(y - b_2)[\lambda L_1 + (\alpha_w + \lambda) L_2] + (r + \lambda + \alpha_w \sigma)\alpha_w \sigma y L_1$$
$$- (r + \lambda + \alpha_w \sigma) b_2 \alpha_w \sigma L_1 - (r + \lambda + \alpha_w \sigma)\alpha_w \sigma y L_1 - (r + \lambda + \alpha_w \sigma)\lambda y L_1$$
$$+ (r + \lambda + \alpha_w \sigma)\lambda b_1 L_1 - r\lambda b_1 L_1 + (\alpha_w \sigma + \lambda) r b_1 L_1 + (\alpha_w \sigma + \lambda)\alpha_w \sigma b_2 L_1$$
$$= (r + \lambda + \alpha_w \sigma)(y - b_2)[\lambda L_1 + (\alpha_w + \lambda) L_2] - r\alpha_w \sigma L_1 b_2 - (r + \lambda + \alpha_w \sigma)\lambda y L_1$$
$$+ (r + \lambda + \alpha_w \sigma)\lambda b_1 L_1 + r\alpha_w \sigma L_1 b_1$$
$$= (r + \lambda + \alpha_w \sigma)\{(y - b_2)[\lambda L_1 + (\alpha_w + \lambda) L_2] - (y - b_1)\lambda L_1\} - r\alpha_w \sigma L_1 (b_2 - b_1)$$

$$\Rightarrow \quad \Pi_2 - \Pi_1 = \frac{(r + \lambda + \alpha_w \sigma)\{(y - b_2)[\lambda L_1 + (\alpha_w + \lambda) L_2] - (y - b_1)\lambda L_1\} - r\alpha_w \sigma L_1 (b_2 - b_1)}{[(\alpha_w \sigma + \lambda) L_1 + (\alpha_w + \lambda) L_2](r + \lambda + \alpha_w \sigma)\frac{r+\lambda}{\alpha_e}}$$

Thus, $\Pi_2 - \Pi_1$ is proportional to:

$$T(\sigma) = (r + \lambda + \alpha_w \sigma)\{(y - b_2)[\lambda L_1 + (\alpha_w + \lambda) L_2] - (y - b_1)\lambda L_1\} - r\alpha_w \sigma L_1 (b_2 - b_1) \tag{75}$$

For an equilibrium, we need one of the following three conditions:

$\sigma = 0$ and $T(0) < 0$; $\sigma = 1$ and $T(1) > 0$; or $\sigma \in (0,1)$ and $T(\sigma) = 0$ (76)

To check when the last condition to hold, we need to evaluate (75) in the limits $\sigma = 0$ and $\sigma = 1$. Suppose $\sigma = 0$ and $T(\sigma) = 0$. Solving (75) for y in this case yields:

$$T(0) = (r+\lambda)\{(y-b_2)[\lambda L_1 + (\alpha_w+\lambda)L_2] - (y-b_1)\lambda L_1\} = 0$$
$$\Rightarrow \quad [\lambda L_1 + (\alpha_w+\lambda)L_2 - \lambda L_1]y = [\lambda L_1 + (\alpha_w+\lambda)L_2]b_2 - \lambda L_1 b_1$$
$$\Rightarrow \quad (\alpha_w+\lambda)L_2 y = (\alpha_w+\lambda)L_2 b_2 + \lambda L_1(b_2-b_1)$$
$$\Rightarrow \quad y = b_2 + \frac{\lambda L_1(b_2-b_1)}{(\alpha_w+\lambda)L_2}$$

Notice that $y < b_2 + \frac{\lambda L_1(b_2-b_1)}{(\alpha_w+\lambda)L_2}$ implies that no firms will offer a wage higher than w_1. Next, suppose $\sigma = 1$ and $T(\sigma) = 0$. Solving (75) for y in this case yields:

$$T(1) = (r+\lambda+\alpha_w)\{(y-b_2)[\lambda L_1 + (\alpha_w+\lambda)L_2] - (y-b_1)\lambda L_1\} - r\alpha_w L_1(b_2-b_1) = 0$$
$$\Rightarrow \quad [\lambda L_1 + (\alpha_w+\lambda)L_2 - \lambda L_1]y = [\lambda L_1 + (\alpha_w+\lambda)L_2]b_2 - \lambda L_1 b_1 + \frac{r\alpha_w L_1(b_2-b_1)}{r+\lambda+\alpha_w}$$
$$\Rightarrow \quad (\alpha_w+\lambda)L_2 y = (\alpha_w+\lambda)L_2 b_2 + \lambda L_1(b_2-b_1) + \frac{r\alpha_w L_1(b_2-b_1)}{r+\lambda+\alpha_w}$$
$$\Rightarrow \quad y = b_2 + \frac{\lambda L_1(b_2-b_1)}{(\alpha_w+\lambda)L_2} + \frac{r\alpha_w L_1(b_2-b_1)}{(r+\lambda+\alpha_w)(\alpha_w+\lambda)L_2}$$

Notice that $y > b_2 + \frac{\lambda L_1(b_2-b_1)}{(\alpha_w+\lambda)L_2} + \frac{r\alpha_w L_1(b_2-b_1)}{(r+\lambda+\alpha_w)(\alpha_w+\lambda)L_2}$ implies that all firms will offer w_2. Combining the two, we see that there exists a unique solution to (76), and that $0 < \sigma < 1$ if and only if $\underline{y} < y < \overline{y}$ where:

$$\underline{y} = b_2 + \frac{\lambda L_1(b_2-b_1)}{(\alpha_w+\lambda)L_2}$$
and (77)
$$\overline{y} = \underline{y} + \frac{r\alpha_w L_1(b_2-b_1)}{(r+\lambda+\alpha_w)(\alpha_w+\lambda)L_2}$$

When productivity is low all firms pay $w_1 = b_1$, when it is high all firms pay $w_2 = b_2$, and when it is intermediate there is wage dispersion. When $\sigma \in (0,1)$, we can solve $T(\sigma) = 0$ for σ and use (72) to solve for w_1 and the distribution of paid wages explicitly. Notice that we get at most two wages here, but the argument can be generalized to many types of workers. Of course, all of this is for given arrival rates, and the value of σ that solves (76) depends on α_w. Using the matching function, we can endogenize:

$$\alpha_w = \frac{m(L_1u_1 + L_2u_2, 1)}{L_1u_1 + L_2u_2} \quad (78)$$

where $L_1u_1 + L_2u_2$ is the number of unemployed workers in the economy and we assume that all firms are freely posting a vacancy, so that $v = 1$, with the idea being that each one will hire as many workers as it can get. Since u_2 depends on σ, so does α_w. An equilibrium is a pair (α_w, σ) satisfying (76) and (78).

6.2 *On-the-Job Search*

In the previous model, firms may pay higher wages to increase the inflow of workers. One can also think of an environment where firms may pay higher wages to reduce the outflow of workers. Now we will look at model where both margins are at work, and firms that pay higher wages both increase the inflow and reduce the outflow of workers. The model builds on the on-the-job search framework in section 3. The arrival rates are α_0 and α_1 while unemployed and employed, and every offer is a random draw from $F(w)$. For ease of presentation, we begin with the case $\alpha_0 = \alpha_1 = \alpha$, which implies $w_R = b$ by (24), and return to the general case later. Since all unemployed workers use a common reservation wage, and clearly no firms post $w < w_R$, the unemployed accept all offers and we have

$$u = \frac{\lambda}{\lambda+\alpha}$$

as a special case of (25). Also, the distribution of paid wages is the special case of (26) where $F(w_R) = 0$:

$$G(w) = \frac{\lambda F(w)}{\lambda+\alpha[1-F(w)]} \tag{79}$$

If a firm posts $w \geq w_R$, a worker he contacts accepts if he is currently unemployed or currently employed but at a lower wage. Thus, the firm will meet an accepting worker with probability $\alpha u + \alpha(1-u)G(w)$. The employment relationship then yields flow profit $y - w$ until the worker leaves either due to an exogenous separation or a better offer, which occurs at a rate $\lambda + \alpha[1-F(w)]$. Therefore, the value of a filled position for firms is:

$$r\Pi(w) = \alpha[u + (1-u)G(w)](y-w) - \{\lambda + \alpha[1-F(w)]\}\Pi(w)$$

Substitute for $u = \frac{\lambda}{\lambda+\alpha}$ and (79) and solve for the expected profit from w:

$$\begin{aligned}
\{r + \lambda + \alpha[1-F(w)]\}\Pi(w) &= \alpha\left[\frac{\lambda}{\lambda+\alpha} + \left(1 - \frac{\lambda}{\lambda+\alpha}\right)\frac{\lambda F(w)}{\lambda+\alpha[1-F(w)]}\right](y-w) \\
&= \left[\frac{1}{\lambda+\alpha} + \frac{\alpha F(w)}{(\lambda+\alpha)\{\lambda+\alpha[1-F(w)]\}}\right]\alpha\lambda(y-w) = \frac{\lambda+\alpha[1-F(w)]+\alpha F(w)}{(\lambda+\alpha)\{\lambda+\alpha[1-F(w)]\}}\alpha\lambda(y-w) \\
&= \frac{\lambda+\alpha}{(\lambda+\alpha)\{\lambda+\alpha[1-F(w)]\}}\alpha\lambda(y-w) = \frac{\alpha\lambda(y-w)}{\lambda+\alpha[1-F(w)]}
\end{aligned}$$

$$\Rightarrow \quad \Pi(w) = \frac{\alpha\lambda(y-w)}{\{\lambda+\alpha[1-F(w)]\}\{r+\lambda+\alpha[1-F(w)]\}} \tag{80}$$

Again, equilibrium requires that any posted wage yields the same profit, which is at least as large as profit from any other wage. Clearly no firm posts $w < w_R = b$ or $w > y$. In fact, one can show that the support of F is $[b, \overline{w}]$ for some $\overline{w} < y$, and there are neither mass points nor gaps on the support. Suppose there were a mass point at some $\widehat{w}$. Then a firm posting $\widehat{w} + \varepsilon$ would be able to hire away any worker it contacts from a $\widehat{w}$ firm, increasing its revenue discretely with only an ε increase in cost, which means $\widehat{w}$ does not maximize profit. Suppose there were a gap in F, say between $\widehat{w}$ and $\widetilde{w}$. Then a firm posting $\widetilde{w}$ could lower its wage and reduce cost without reducing its inflow or increasing its outflow of workers, and again $\widetilde{w}$ could not maximize profit. We now construct F explicitly. The key observation is that firms earn equal profit from all posted wages, including the lowest $w = b$. Thus, $\Pi(b) = \Pi(w)$ for all $w \in [b, \overline{w}]$. Since $F(b) = 0$, we can rewrite (80) in this case to:

$$\Pi(b) = \frac{\alpha\lambda(y-b)}{(\lambda+\alpha)(r+\lambda+\alpha)}$$

Using the indifference condition $\Pi(b) = \Pi(w)$, we can combine the equation above and (80) to get:

$$\frac{\alpha\lambda(y-b)}{(\alpha+\lambda)^2} = \frac{\alpha\lambda(y-w)}{\{\lambda+\alpha[1-F(w)]\}^2}$$

where $r \approx 0$ has been imposed. Rewriting yields:

$$\{\lambda + \alpha[1 - F(w)]\}^2 = (\alpha + \lambda)^2 \frac{y-w}{y-b}$$

$$\Rightarrow \quad \lambda + \alpha[1 - F(w)] = (\alpha + \lambda)\sqrt{\frac{y-w}{y-b}}$$

$$\Rightarrow \quad \alpha F(w) = \alpha + \lambda - (\alpha + \lambda)\sqrt{\frac{y-w}{y-b}} = (\alpha + \lambda)\left(1 - \sqrt{\frac{y-w}{y-b}}\right)$$

$$\Rightarrow \quad F(w) = \frac{\lambda+\alpha}{\alpha}\left(1 - \sqrt{\frac{y-w}{y-b}}\right) \tag{81}$$

We know that the lower bound is b, and the upper bound $\overline{w}$ can be found by solving $F(\overline{w}) = 1$. This yields the unique distribution consistent with equal profit for all wages posted. In words, the outcome is as follows. All unemployed workers accept the first offer they receive, and move up the ladder each time a better offer comes along, but also return to unemployment periodically due to exogenous layoffs. There is a nondegenerate distribution of wages posted by firms F given by (81), and of wages earned by workers G, given by inserting F into (79). The model is consistent with many observations concerning labor turnover, and also concerning firms, e.g. the fact that high wage firms are bigger. There are many interesting extensions, including the $\alpha_0 \neq \alpha_1$. In this case (25) becomes:

$$u = \frac{\lambda}{\lambda+\alpha_0[1-F(w_R)]} = \frac{\lambda}{\lambda+\alpha_0}$$

since no firms offer a wage lower than w_R. As before we look at the steady state where flows into and out of employment are equal. Given $\alpha_0 \neq \alpha_1$ equation (26) continues to hold, except that $F(w_R) = 0$ implies:

$$G(w) = \frac{\lambda F(w)}{\lambda+\alpha_1[1-F(w)]}$$

A firm that posts $w \geq w_R$ will now get a worker with probability $\alpha_0 u + \alpha_1(1-u)G(w)$, while the worker still leaves at a rate $\lambda + \alpha_1[1-F(w)]$. Therefore, the value function for firms becomes:

$$r\Pi(w) = [\alpha_0 u + \alpha_1(1-u)G(w)](y-w) - \{\lambda + \alpha_1[1-F(w)]\}\Pi(w)$$

Using the same procedure as before, the expected profit from w emerges:

$$\begin{aligned}\{r + \lambda + \alpha_1[1-F(w)]\}\Pi(w) &= \left[\frac{\alpha_0\lambda}{\lambda+\alpha_0} + \alpha_1\left(1-\frac{\lambda}{\lambda+\alpha_0}\right)\frac{\lambda F(w)}{\lambda+\alpha_1[1-F(w)]}\right](y-w)\\ &= \left[\frac{\alpha_0}{\lambda+\alpha_0} + \frac{\alpha_0\alpha_1 F(w)}{(\lambda+\alpha_0)\{\lambda+\alpha_1[1-F(w)]\}}\right]\lambda(y-w)\\ &= \frac{\alpha_0\{\lambda+\alpha_1[1-F(w)]\}+\alpha_0\alpha_1F(w)}{(\lambda+\alpha_0)\{\lambda+\alpha_1[1-F(w)]\}}\lambda(y-w)\end{aligned}$$

$$\Pi(w) = \frac{\alpha_0(\lambda+\alpha_1)}{(\lambda+\alpha_0)\{\lambda+\alpha_1[1-F(w)]\}\{r+\lambda+\alpha_1[1-F(w)]\}}\lambda(y-w)$$

Evaluated at $w = w_R$, this becomes:

$$\Pi(w_R) = \frac{\alpha_0\lambda(y-w_R)}{(\lambda+\alpha_0)(r+\lambda+\alpha_1)}$$

Next, assume that $r \approx 0$ and equate the two equations using the indifference condition $\Pi(w) = \Pi(w_R)$. Then solve for $F(w)$:

$$\frac{\alpha_0(\lambda+\alpha_1)\lambda(y-w)}{(\lambda+\alpha_0)\{\lambda+\alpha_1[1-F(w)]\}^2} = \frac{\alpha_0\lambda(y-w_R)}{(\lambda+\alpha_0)(\lambda+\alpha_1)}$$

$$\Rightarrow \quad \frac{(\lambda+\alpha_1)(y-w)}{\{\lambda+\alpha_1[1-F(w)]\}^2} = \frac{y-w_R}{\lambda+\alpha_1}$$

$$\Rightarrow \quad \{\lambda+\alpha_1[1-F(w)]\}^2 = (\lambda+\alpha_1)^2\frac{y-w}{y-w_R}$$

$$\Rightarrow \quad \lambda+\alpha_1[1-F(w)] = (\lambda+\alpha_1)\sqrt{\frac{y-w}{y-w_R}}$$

$$\Rightarrow \quad \alpha_1 F(w) = \lambda+\alpha_1-(\lambda+\alpha_1)\sqrt{\frac{y-w}{y-w_R}}$$

$$\Rightarrow \quad F(w) = \frac{\lambda+\alpha_1}{\alpha_1}\left(1-\sqrt{\frac{y-w}{y-w_R}}\right) \tag{82}$$

To determine w_R, we first insert (82) into (24):

$$w_R = b + (\alpha_0 - \alpha_1) \int_{w_R}^{\infty} \frac{1 - F(w)}{\lambda + \alpha_1[1 - F(w)]} dw = b + (\alpha_0 - \alpha_1) \int_{w_R}^{\infty} \frac{1 - \frac{\lambda + \alpha_1}{\alpha_1}\left(1 - \sqrt{\frac{y - w}{y - w_R}}\right)}{\lambda + \alpha_1\left[1 - \frac{\lambda + \alpha_1}{\alpha_1}\left(1 - \sqrt{\frac{y - w}{y - w_R}}\right)\right]} dw$$

$$= b + (\alpha_0 - \alpha_1) \int_{w_R}^{\infty} \frac{-\frac{\lambda}{\alpha_1} + \frac{\lambda + \alpha_1}{\alpha_1}\sqrt{\frac{y - w}{y - w_R}}}{\lambda + \alpha_1\left[1 - \left(1 - \sqrt{\frac{y - w}{y - w_R}}\right)\right]} dw$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1} \int_{w_R}^{\infty} \frac{-\lambda + (\lambda + \alpha_1)\sqrt{\frac{y - w}{y - w_R}}}{(\lambda + \alpha_1)\sqrt{\frac{y - w}{y - w_R}}} dw$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1} \int_{w_R}^{\infty} \left[1 - \frac{\lambda}{\lambda + \alpha_1}\left(\frac{y - w}{y - w_R}\right)^{-\frac{1}{2}}\right] dw$$

Integration yields:

$$w_R = b + \frac{\alpha_0 - \alpha_1}{\alpha_1} \left| w + \frac{2\lambda}{\lambda + \alpha_1}\left(\frac{y - w}{y - w_R}\right)^{\frac{1}{2}} (y - w_R) \right|_{w_R}^{\overline{w}}$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1} \left[\overline{w} + \frac{2\lambda}{\lambda + \alpha_1}\left(\frac{y - \overline{w}}{y - w_R}\right)^{\frac{1}{2}} (y - w_R) - w_R - \frac{2\lambda}{\lambda + \alpha_1}(y - w_R)\right]$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1} \left\{\overline{w} - w_R + \frac{2\lambda(y - w_R)}{\lambda + \alpha_1}\left[\left(\frac{y - \overline{w}}{y - w_R}\right)^{\frac{1}{2}} - 1\right]\right\}$$

To proceed, note that $F(\overline{w}) = 1$ as no firms offer higher wages than $\overline{w}$. Thus, (82) can be written as:

$$1 = \frac{\lambda + \alpha_1}{\alpha_1}\left(1 - \sqrt{\frac{y - \overline{w}}{y - w_R}}\right)$$

$$\Rightarrow \quad \sqrt{\frac{y - \overline{w}}{y - w_R}} = 1 - \frac{\alpha_1}{\lambda + \alpha_1} = \frac{\lambda}{\lambda + \alpha_1}$$

$$\Rightarrow \quad y - \overline{w} = \left(\frac{\lambda}{\lambda + \alpha_1}\right)^2 (y - w_R) > 0$$

where the inequality holds as long as $y > w_R$. Insert this into the equation for w_R:

$$w_R = b + \frac{\alpha_0 - \alpha_1}{\alpha_1}\left(y - \left(\frac{\lambda}{\lambda + \alpha_1}\right)^2 (y - w_R) - w_R + \frac{2\lambda(y - w_R)}{\lambda + \alpha_1}\left\{\left[\frac{\left(\frac{\lambda}{\lambda + \alpha_1}\right)^2 (y - w_R)}{y - w_R}\right]^{\frac{1}{2}} - 1\right\}\right)$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1}\left[y - w_R - \left(\frac{\lambda}{\lambda + \alpha_1}\right)^2 (y - w_R) + \frac{2\lambda(y - w_R)}{\lambda + \alpha_1}\left(\frac{\lambda}{\lambda + \alpha_1} - 1\right)\right]$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1}\left[1 - \left(\frac{\lambda}{\lambda + \alpha_1}\right)^2 - \frac{2\lambda}{\lambda + \alpha_1}\frac{\alpha_1}{\lambda + \alpha_1}\right](y - w_R)$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1}\frac{(\lambda + \alpha_1)^2 - \lambda^2 - 2\lambda\alpha_1}{(\lambda + \alpha_1)^2}(y - w_R)$$

$$= b + \frac{\alpha_0 - \alpha_1}{\alpha_1}\frac{\lambda^2 + 2\lambda\alpha_1 + \alpha_1^2 - \lambda^2 - 2\lambda\alpha_1}{(\lambda + \alpha_1)^2}(y - w_R) = b + (\alpha_0 - \alpha_1)\frac{\alpha_1}{(\lambda + \alpha_1)^2}(y - w_R)$$

$$= \frac{(\lambda + \alpha_1)^2 b + (\alpha_0 - \alpha_1)\alpha_1}{(\lambda + \alpha_1)^2}(y - w_R)$$

Finally, solving for w_R we get:

$$\frac{(\lambda+\alpha_1)^2+(\alpha_0-\alpha_1)\alpha_1}{(\lambda+\alpha_1)^2} w_R = \frac{(\lambda+\alpha_1)^2 b+(\alpha_0-\alpha_1)\alpha_1 y}{(\lambda+\alpha_1)^2}$$

$$\Rightarrow \quad w_R = \frac{(\lambda+\alpha_1)^2 b+(\alpha_0-\alpha_1)\alpha_1 y}{(\lambda+\alpha_1)^2+(\alpha_0-\alpha_1)\alpha_1} \tag{83}$$

One can check that in the limit as $\alpha_1 \to 0$, $\overline{w} = w_R$, which means there is a single wage, $w = w_R = b$. This is the Diamond-result as a special case when there is no on-the-job search. Also, in the limit as $\alpha_1 \to \infty$, $\overline{w} = y$ and $G(w) = 0$ for all $w < y$. Then all workers earn $w = y$. Moreover, as $\alpha_1 \to \infty$, clearly $u \to 0$. Hence, the competitive solution also emerges as a special case when α_0 and α_1 get large.

7. *Efficiency*

7.1 A One-Shot Model

TBA

7.2 A Dynamic Model

TBA